Subproduct systems and superproduct systems (or: behind the scenes of the dilation theory of CP-semigroups)

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Technion

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This talk is part of my joint work in progress with **Michael Skeide**

The objects of study

$\mathbb{S}$ a semigroup of $\mathbb{R}_+^k$, such that $0 \in \mathbb{S}$.
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  1. T_s is a (contractive) CP map for all s,
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(**UCP**)	$t \mapsto T_t(a)$	evolution in an irreversible quantum system
($*$**auto**)	$t \mapsto \alpha_t(a)$	evolution in a reversible quantum system

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Example

If $T_1, \ldots, T_k$ are k commuting CP maps, then we get a CP-semigroup $(T_s)_{s \in \mathbb{N}^k}$ over $\mathbb{S} = \mathbb{N}^k$:

$$T_s = T_1^{s_1} \circ \cdots \circ T_k^{s_k} \quad \text{where } s = (s_1, \ldots, s_k) \in \mathbb{N}^k.$$

Every CP-semigroup over $\mathbb{S} = \mathbb{N}^k$ arises this way.

Bhat's dilation theorem[1]

Theorem (Bhat, 1996)

Let $T = (T_t)_{t\geq 0}$ be a CP-semigroup on $\mathcal{B}(H)$. Then there exists a Hilbert space K containing H, and an E-semigroup $\vartheta = (\vartheta_t)_{t\geq 0}$ on $\mathcal{B}(K)$, such that

$$T_t(A) = P_H \vartheta_t(A) P_H \quad , \qquad \textit{for all } t \geq 0 \textit{ and } A \in \mathcal{B}(H).$$

$$\begin{tikzcd}
\mathcal{B}(K) \arrow[r, "\vartheta_t"] & \mathcal{B}(K) \arrow[d, "P_H \bullet P_H"] \\
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[1] Result also works for $\mathbb{N}$ instead of $\mathbb{R}_+$ (see abstract).

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$$T_t(P_H A P_H) = P_H \vartheta_t(A) P_H \quad , \qquad \textit{for all } t \geq 0 \textit{ and } A \in \mathcal{B}(K).$$

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Remark

Different notions of dilations of CP-semigroups have been studied since 70s: Davies, Evans-Lewis, Hudson-Parthasarathy, Kummerer, Sauvageout ...

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For the presentation, narrow the scope:
Let $T = (T_s)_{s \in \mathbb{S}}$ be a CP-semigroup over $\mathbb{S} \subseteq \mathbb{R}_+^k$, acting on a **von Neumann algebra** $\mathcal{B}$, such that every T_s is a **normal map.**

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Definition

A **dilation** of T is a triple $(\mathcal{A}, \vartheta, p)$, where $\mathcal{A}$ is a von Neumann algebra, $\vartheta = (\vartheta_s)_{s\in\mathbb{S}}$ is a semigroup of normal *-endomorphism, and $p \in \mathcal{A}$ is a projection, such that $\mathcal{B} = p\mathcal{A}p$, and such that

$$T_s(b) = p\vartheta_s(b)p \quad \text{for all } b \in \mathcal{B}, s \in \mathbb{S}.$$

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Arveson, Bhat, Bhat-Skeide, Markiewicz, Muhly-Solel, Powers, SeLegue, S., S.-Solel, Solel, Vernik,. . .

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Questions

1. **Find necessary & sufficient conditions for existence of dilation.**
2. **For fixed k, does every CP-semigroup over $\mathbb{N}^k$ have a dilation?**

.

The GNS representation $(\mathcal{E}, \xi)$ of a CP map

Let $T : \mathcal{B} \to \mathcal{B}$ be a CP map. Then there exists a unique W*-correspondence[2] $\mathcal{E}$ over $\mathcal{B}$, and a vector $\xi \in \mathcal{E}$, such that

$$\operatorname{span} \overline{\mathcal{B}\xi\mathcal{B}}^{s} = \mathcal{E}$$

and

$$\langle \xi, b\xi \rangle = T(b) \quad \text{for all } b \in \mathcal{B}.$$

[2] A bimodule over $\mathcal{B}$, that has a $\mathcal{B}$-valued inner product. Equivalently, one may use Skeide's **von Neumann modules** (and we do).

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Construction: on $\mathcal{E}_0 = \mathcal{B} \otimes \mathcal{B}$ put inner product

$$\langle a \otimes b, c \otimes d \rangle = b^* T(a^* c) d$$

and bimodule operation

$$a(x \otimes y)d = ax \otimes yd.$$

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by

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and then extend linearly.

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$$= b^* T_t(T_s(a^* a)) b = b^* T_{t+s}(a^* a) b = \langle a\xi_{s+t} b, a\xi_{s+t} b \rangle.$$

$w_{s,t}$ **is an isometry!**

Subproduct systems[4]

Definition

A **subproduct system** is a family $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s \in \mathbb{S}}$ of $\mathcal{B}$-correspondences, together with a family $\{w_{s,t} : \mathcal{E}_{s+t} \to \mathcal{E}_s \odot \mathcal{E}_t\}$ of isometric bimodule maps, which iterate associatively

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A **product system** is a subproduct system in which $w_{s,t}$ are all unitaries.

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Subproduct systems[4]

Definition

A **subproduct system** is a family $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s\in\mathbb{S}}$ of $\mathcal{B}$-correspondences, together with a family $\{w_{s,t} : \mathcal{E}_{s+t} \to \mathcal{E}_s \odot \mathcal{E}_t\}$ of isometric bimodule maps, which iterate associatively, i.e., the following diagram is commutative $(\forall r, s, t)$:

$$\begin{tikzcd}
\mathcal{E}_{r+s+t} \arrow[r] \arrow[d] & \mathcal{E}_r \odot \mathcal{E}_{s+t} \arrow[d] \\
\mathcal{E}_{r+s} \odot \mathcal{E}_t \arrow[r] & \mathcal{E}_r \odot \mathcal{E}_s \odot \mathcal{E}_t
\end{tikzcd}$$

A **product system** is a subproduct system in which $w_{s,t}$ are all unitaries.

Definition

A family $\{\xi_s \in \mathcal{E}_s\}_{s\in\mathbb{S}}$ is called a **unit** if $w_{s,t}\xi_{s+t} = \xi_s \odot \xi_t$ for all s, t.

[4] **Inclusion systems** by Bhat-Mukherjee.

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For every CP-semigroup on $\mathcal{B}$, there exists a subproduct system $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s\in\mathbb{S}}$ of $\mathcal{B}$-correspondences (called the **GNS subproduct system**) and a unit $(\xi_s)_{s\in\mathbb{S}}$ such that

$$T_s(b) = \langle \xi_s, b\xi_s \rangle \quad \text{for all } s \in \mathbb{S}, b \in \mathcal{B}.$$

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Let T be a Markov semigroup. If the GNS subproduct system of T can be embedded in a **product system**, *then T has a unital dilation $(\mathcal{A}, \vartheta, p)$. In fact, one can take $\mathcal{A} = \mathscr{B}^a(E)$, where E is some (full) $\mathcal{B}$-correspondence.*

Markov semigroup = unital CP-semigroup.

An application

Theorem (S.-Skeide, see also Bhat 98, Solel 2006)

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$$T_1^{n_1} \circ T_2^{n_2}(b) = p\vartheta_1^{n_1} \circ \vartheta_2^{n_2}(b)p \quad \textit{for all } b \in \mathcal{B}, n_1, n_2 \in \mathbb{N}.$$

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Remark: In fact we have $\mathcal{A} = \overline{\mathcal{A}p\mathcal{A}}^s = \mathcal{B}^a(E)$, where $E = \overline{\mathcal{A}p}^s$. In particular, $\mathcal{A}$ is **Morita equivalent** to $\mathcal{B}$ (in the sense of Rieffel).

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4. What about dilations $(\mathcal{A}, \vartheta, p)$, where $\mathcal{A} \neq \mathcal{B}^a(E)$?

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[S.-Solel] construct a subproduct system over $\mathbb{N}^3$ that cannot be embedded into a product system. We apply the above theorem to that subproduct system.

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Problem: this does not rule out the existence of **non-minimal** dilations.

Minimality

Let $T = (T_s)_{s \in \mathbb{S}}$ be a CP-semigroup over $\mathbb{S}$, and $(\mathcal{A}, \vartheta, p)$ a dilation. Suppose that $\mathcal{B} \subseteq \mathcal{B}(H)$ and that $\mathcal{A} \subseteq \mathcal{B}(K)$, so that $p = P_H$.

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2. "Spatial minimality", that is, $\mathcal{A} = \overline{\mathcal{A} p \mathcal{A}}^s$. Assuming 1, same as:
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3. "Incompressibility": there is no nontrivial projection $p \leq q \in \mathcal{A}$ s.t.

$$q\vartheta_s(\cdot)q : q\mathcal{A}q \to q\mathcal{A}q \quad , \quad q\vartheta_s(\cdot)q : qaq \mapsto q\vartheta_s(qaq)q,$$

is an E-semigroup, and a dilation of T.

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We have an example of a dilation $(\mathcal{A}, \vartheta, p)$ over $\mathbb{N}^2$, which satisfies 2, but not 1. After restricting to $W^*(\cup_{s\in\mathbb{S}}\vartheta_s(\mathcal{B}))$, and then compressing to the minimal compressing q, one obtains an algebraically minimal and incompressible dilation (1+3), which does **not** satisfy 2.

Dilation $\Rightarrow$ what?

Let $T = (T_s)_{s \in \mathbb{S}}$ be a CP-semigroup on $\mathcal{B}$, and $(\mathcal{A}, \vartheta, p)$ a dilation. Following a construction from [Skeide02], we see what structure arises.

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Let $T = (T_s)_{s \in \mathbb{S}}$ be a CP-semigroup on $\mathcal{B}$, and $(\mathcal{A}, \vartheta, p)$ a dilation. We constructed a family $(E_s)_{s \in \mathbb{S}}$ of $\mathcal{B}$-corresopndences, and a family $(\eta_s)_{s \in \mathbb{S}}$ of unit vectors ($\eta_s \in E_s$) that represent T:

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Q: is $(E_s)_{s \in \mathbb{S}}$ a PRODUCT system?

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Define

$$v_{s,t} : E_s \odot E_t \to E_{s+t} \quad \left(\text{ really } E_s \overline{\odot}^s E_t \right)$$

$$v_{s,t} : x_s \odot y_t \mapsto \vartheta_t(x_s)y_t \,.$$

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A direct calculation shows:

$$\langle x_s \odot y_t, x_s' \odot y_t' \rangle = \ldots = \langle \vartheta_t(x_s) y_t, \vartheta_t(x_s') y_t' \rangle.$$

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$$E_s \odot E_t \subseteq E_{s+t}.$$

$(E_s)_{s \in \mathbb{S}}$ is a **superproduct system** (but not always a product system).

Superproduct systems

Definition

A **superproduct system** is a family $E^{\otimes} = (E_s)_{s \in \mathbb{S}}$ of $\mathcal{B}$-correspondences, together with a family $\{v_{s,t} : E_s \odot E_t \to E_{s+t}\}$ of isometric bimodule maps, which iterate associatively

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$$\begin{tikzcd} E_r \odot E_s \odot E_t \arrow[r] \arrow[d] & E_r \odot E_{s+t} \arrow[d] \\ E_{r+s} \odot E_t \arrow[r] & E_{r+s+t} \end{tikzcd}$$

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A **product system** is a superproduct system in which $v_{s,t}$ are all unitaries.

Recap

Subproduct system:	$\mathcal{E}_s \odot \mathcal{E}_t \supseteq \mathcal{E}_{s+t}$
Product system:	$E_s \odot E_t = E_{s+t}$
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For every CP-semigroup T on $\mathcal{B}$, there exists a subproduct system $\mathcal{E}^{\circledcirc} = (\mathcal{E}_s)_{s\in\mathbb{S}}$ of $\mathcal{B}$-correspondences (the **GNS subproduct system**) and a unit $(\xi_s)_{s\in\mathbb{S}}$ such that

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If T has a dilation $(\mathcal{A}, \vartheta, p)$, then the GNS subproduct system must **embed into a superproduct system.**

Dilations and superproduct systems

Theorem (S.-Skeide)

Let $T = (T_s)_{s \in \mathbb{S}}$ be a Markov semigroup on a von Neumann algebra $\mathcal{B}$.

- *A sufficient condition for T to a have a dilation, is that the GNS subproduct system of T embeds into a* **product** *system.*
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We have an example of a subproduct system over $\mathbb{N}^3$ that cannot be embedded into a **superproduct** system.

The truth: the SPS is not the GNS subproduct system of a CP-semigroup, so the proof does not really go like that ...

Another way subproduct systems arise

Let E be a full W*-correspondence over $\mathcal{B}$, and $\mathscr{B}^a(E)$ the **adjointable** operators on E. E is a **Morita W* equivalence** from $\mathscr{B}^a(E)$ to $\mathcal{B}$:

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Theorem (S.-Skeide, see also S.-Solel)

Every subproduct system over $\mathcal{B}$ is the subproduct system of $\mathcal{B}$-correspondences associated with some normal CP-semigroup T acting on some $\mathscr{B}^a(E)$, where E is a $\mathcal{B}$-correspondence.

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Morita equivalence behaves nicely w.r.t. inclusions into product systems.

Thank you!